

# PROBABILISTIC GLOBAL WELL-POSEDNESS OF THE ENERGY-CRITICAL DEFOCUSING QUINTIC NONLINEAR WAVE EQUATION ON $\mathbb{R}^3$

TADAHIRO OH AND OANA POCOVNICU

**ABSTRACT.** We prove almost sure global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on  $\mathbb{R}^3$  with random initial data in  $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$  for  $s > \frac{1}{2}$ . The main new ingredient is a uniform probabilistic energy bound for approximating random solutions.

**RÉSUMÉ.** On considère l'équation des ondes critique défocalisante dans  $\mathbb{R}^3$  à données initiales aléatoires dans  $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ , avec  $s > \frac{1}{2}$ . On établit que ce problème est globalement bien-posé presque sûrement. Le principal ingrédient nouveau de la preuve est une estimation probabiliste uniforme de l'énergie des solutions approchées.

## 1. INTRODUCTION

**1.1. Nonlinear wave equation.** We consider the Cauchy problem for the energy-critical defocusing quintic nonlinear wave equation (NLW) on  $\mathbb{R}^3$ :

$$\begin{cases} \partial_t^2 u - \Delta u + u^5 = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \quad (1.1)$$

where  $u$  is a real-valued function. NLW has been studied extensively from both applied and theoretical points of view, in particular in three spatial dimensions due to its physical importance. In this paper, we study the global-in-time behavior of solutions to (1.1) with *random* and *rough* initial data below the energy space.

It is well known that the quintic NLW (1.1) on  $\mathbb{R}^3$  is invariant under the following dilation symmetry:

$$u(t, x) \mapsto u_\lambda(t, x) := \lambda^{\frac{1}{2}} u(\lambda t, \lambda x). \quad (1.2)$$

Namely, if  $u$  is a solution to (1.1), then  $u_\lambda$  is also a solution to (1.1) with rescaled initial data. Recall that the  $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ -norm is invariant under this dilation symmetry:

$$\|(u_\lambda(0), \partial_t u_\lambda(0))\|_{\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} = \|(u(0), \partial_t u(0))\|_{\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)}.$$

Moreover, the conserved energy  $E(u)$  defined by

$$E(u) = E(u, \partial_t u) := \int_{\mathbb{R}^3} \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{6} u^6 dx \quad (1.3)$$

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is also invariant under the dilation symmetry (1.2). This explains why the quintic NLW on  $\mathbb{R}^3$  is called *energy-critical*. In view of Sobolev's inequality:  $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ , we see that  $E(u, \partial_t u) < \infty$  if and only if

$$(u, \partial_t u) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3).$$

In the following, we refer to  $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$  as the energy space.

Let us briefly recall the known results on global well-posedness of the defocusing NLW in the energy space. For an energy-subcritical defocusing NLW on  $\mathbb{R}^3$  with nonlinearity  $|u|^{p-1}u$ ,  $p < 5$ , the conservation of the energy allows us to iterate the local-in-time argument and obtain global well-posedness in  $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ . The energy-critical defocusing quintic NLW (1.1) on  $\mathbb{R}^3$ , however, lies at a rather delicate balance of dispersion by the linear evolution and concentration due to the nonlinearity, and the issue of global well-posedness for (1.1) is more intricate. After substantial efforts made by many mathematicians, it is now known that (1.1) is globally well-posed in the energy space and all finite energy solutions scatter [28, 15, 16, 26, 27, 18, 14, 2, 1, 29]. Lastly, recall that the energy-critical quintic NLW (1.1) on  $\mathbb{R}^3$  is known to be ill-posed below the energy space [11].

Recently, there has been a significant development in probabilistic construction of local-in-time and global-in-time solutions to hyperbolic and dispersive PDEs below certain regularity thresholds (such as a scaling critical regularity), where the equations are known to be ill-posed deterministically. In particular, following the methodology developed in [6, 9, 4, 25], one can easily prove almost sure local well-posedness of (1.1) below the energy space (Theorem 1.3). Therefore, it is natural to study the long time behavior of such local solutions constructed in a probabilistic manner.

Our main goal in this paper is to prove almost sure global well-posedness of (1.1) below the energy space under suitable randomization of initial data. See Theorem 1.5 below. In particular, this settles the question of almost sure global well-posedness for *large*, random, and rough initial data, in the physically important case of the energy-critical NLW on  $\mathbb{R}^3$ . This case was not addressed in the previous works on the subject. Indeed, previously, Lührmann-Mendelson [23] proved almost sure global well-posedness below the scaling critical regularity for energy-subcritical (sub-quintic) NLW on  $\mathbb{R}^3$ . In the same paper, they also proved almost sure small data global well-posedness for the energy-critical NLW (1.1) on  $\mathbb{R}^3$ . We point out that the methods used in [23] are specific to the energy-subcritical or small data setting and are not applicable to our problem. In fact, a new method was needed for studying the global behavior of solutions to an energy-critical equation with large random initial data. Recently, the second author [25] successfully implemented such a method and proved almost sure global well-posedness below the energy space for the energy-critical NLW on  $\mathbb{R}^d$ ,  $d = 4, 5$ , with large random initial data. The argument in [25], however, fails in the case of  $d = 3$ , and thus we need to develop additional new ideas and perform a more intricate analysis to treat (1.1) on  $\mathbb{R}^3$ .

**1.2. Wiener randomization.** In this subsection, we discuss the randomization for functions on  $\mathbb{R}^3$  that we employ for our main result.

Following the works of Bourgain [6] and Burq-Tzvetkov [9], there have been many results on probabilistic construction of solutions to evolution equations via randomization of initial data. On a compact manifold  $M$ , there is a countable (orthonormal) basis  $\{e_n\}_{n \in \mathbb{N}}$  of  $L^2(M)$  consisting of eigenfunctions of the Laplace-Beltrami operator. This gives a natural way to introduce a randomization as follows. Given  $u_0 = \sum_{n=1}^{\infty} \hat{u}_n e_n \in H^s(M)$ , we can define its randomization  $u_0^\omega$  by

$$u_0^\omega := \sum_{n=1}^{\infty} g_n(\omega) \hat{u}_n e_n, \quad (1.4)$$

where  $\{g_n\}_{n \in \mathbb{N}}$  is a sequence of independent mean zero random variables, satisfying certain moment estimates. When  $M = \mathbb{T}^d$ , we can express  $u_0^\omega$  in (1.4) as

$$u_0^\omega = \Xi(\omega) * u_0, \quad (1.5)$$

where  $\Xi$  is a random distribution given by<sup>1</sup>

$$\Xi(\omega) = \sum_{n \in \mathbb{Z}^d} g_n(\omega) e^{2\pi i n \cdot x}. \quad (1.6)$$

In particular, if  $\{g_n\}_{n \in \mathbb{Z}^d}$  is a sequence of independent standard Gaussian random variables, then  $\Xi$  in (1.6) corresponds to the (mean zero Gaussian) white noise on  $\mathbb{T}^d$ . In this case, we can call the randomization  $u_0^\omega$  given by (1.4) and (1.5) the white noise randomization of  $u_0$ . See Remark 1.2 below.

On the Euclidean space  $\mathbb{R}^d$ , however, there is no countable basis of  $L^2(\mathbb{R}^d)$  consisting of eigenfunctions of the Laplacian and thus there is no ‘natural’ way to introduce a randomization of functions as in (1.4). Randomizations for functions on  $\mathbb{R}^d$  have been considered with respect to some other countable bases of  $L^2(\mathbb{R}^d)$  such as a countable basis of the eigenfunctions of the Laplacian with a confining potential, for example, the harmonic oscillator  $-\Delta + |x|^2$ , [31, 8]. In the following, however, we consider a simple randomization for functions on  $\mathbb{R}^d$ , naturally associated to the Wiener decomposition of the frequency space  $\mathbb{R}_\xi^d$ . See also [23, 4, 5].

Let  $Q_n$  be the unit cube  $Q_n := n + [-\frac{1}{2}, \frac{1}{2})^d$  centered at  $n \in \mathbb{Z}^d$ . For simplicity, we set  $Q := Q_0$ . The Wiener decomposition [32] of the frequency space  $\mathbb{R}_\xi^d$  is given by the uniform partition:  $\mathbb{R}^d = \bigcup_{n \in \mathbb{Z}^d} Q_n$ . Clearly, given a function  $u$  on  $\mathbb{R}^d$ , we have

$$u = \sum_{n \in \mathbb{Z}^d} \chi_{Q_n}(D) u = \sum_{n \in \mathbb{Z}^d} \chi_Q(D - n) u. \quad (1.7)$$

Here,  $\chi_{Q_n}(D)$  denotes the Fourier multiplier operator with symbol  $\chi_{Q_n}$ .

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1. On  $\mathbb{T}^d$ , we have  $e^{2\pi i n \cdot x} = \phi(D - n)\delta$ ,  $n \in \mathbb{Z}^d$ , where  $\phi = \chi_{B(0, \frac{1}{2})}$ . Then,  $\Xi$  in (1.6) can be written as

$$\Xi(\omega) = \sum_{n \in \mathbb{Z}^d} g_n(\omega) \phi(D - n)\delta.$$

Compare this with (1.11) below.

Next, we consider the smoothed version of the decomposition (1.7). Let  $\psi \in \mathcal{S}(\mathbb{R}^d)$  be such that  $\text{supp } \psi \subset [-1, 1]^d$ ,  $\psi(-\xi) = \overline{\psi(\xi)}$ , and

$$\sum_{n \in \mathbb{Z}^d} \psi(\xi - n) \equiv 1 \quad \text{for all } \xi \in \mathbb{R}^d.$$

Then, any function  $u$  on  $\mathbb{R}^d$  can be written as

$$u = \sum_{n \in \mathbb{Z}^d} \psi(D - n)u, \quad (1.8)$$

where  $\psi(D - n)$  denotes the Fourier multiplier operator with symbol  $\psi(\cdot - n)$ .

We now introduce a randomization adapted to the uniform decomposition (1.8). For  $j = 0, 1$ , let  $\{g_{n,j}\}_{n \in \mathbb{Z}^d}$  be a sequence of mean zero complex-valued random variables on a probability space  $(\Omega, \mathcal{F}, P)$  such that  $g_{-n,j} = \overline{g_{n,j}}$  for all  $n \in \mathbb{Z}^d$ ,  $j = 0, 1$ . In particular,  $g_{0,j}$  is real-valued. Moreover, we assume that  $\{g_{0,j}, \text{Re } g_{n,j}, \text{Im } g_{n,j}\}_{n \in \mathcal{I}, j=0,1}$  are independent, where the index set  $\mathcal{I}$  is defined by

$$\mathcal{I} := \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+ \times \{0\}^{d-k-1}. \quad (1.9)$$

Note that  $\mathbb{Z}^d = \mathcal{I} \cup (-\mathcal{I}) \cup \{0\}$ . Then, given a pair  $(u_0, u_1)$  of functions on  $\mathbb{R}^d$ , we define the *Wiener randomization*  $(u_0^\omega, u_1^\omega)$  of  $(u_0, u_1)$  by

$$\begin{aligned} (u_0^\omega, u_1^\omega) &:= (\Xi_0(\omega) * u_0, \Xi_1(\omega) * u_1) \\ &= \left( \sum_{n \in \mathbb{Z}^d} g_{n,0}(\omega) \psi(D - n)u_0, \sum_{n \in \mathbb{Z}^d} g_{n,1}(\omega) \psi(D - n)u_1 \right). \end{aligned} \quad (1.10)$$

Here,  $\Xi_0$  and  $\Xi_1$  are random distributions given by

$$\Xi_j(\omega) = \sum_{n \in \mathbb{Z}^d} g_{n,j}(\omega) \psi(D - n) \delta, \quad j = 0, 1, \quad (1.11)$$

where  $\delta$  denotes the Dirac delta distribution. Note that, if  $u_0$  and  $u_1$  are real-valued, then their randomizations  $u_0^\omega$  and  $u_1^\omega$  defined in (1.10) are also real-valued.

We make the following assumption on the probability distributions  $\mu_{n,j}$  for  $g_{n,j}$ ; there exists  $c > 0$  such that

$$\int e^{\gamma \cdot x} d\mu_{n,j}(x) \leq e^{c|\gamma|^2}, \quad j = 0, 1, \quad (1.12)$$

for all  $n \in \mathbb{Z}^d$ , (i) all  $\gamma \in \mathbb{R}$  when  $n = 0$ , and (ii) all  $\gamma \in \mathbb{R}^2$  when  $n \in \mathbb{Z}^d \setminus \{0\}$ . Note that (1.12) is satisfied by standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

It is easy to see that, if  $(u_0, u_1) \in H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$  for some  $s \in \mathbb{R}$ , then the Wiener randomization  $(u_0^\omega, u_1^\omega)$  is almost surely in  $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ . Note that, under some non-degeneracy condition on the random variables  $\{g_{n,j}\}$ , there is almost surely no gain from randomization in terms of differentiability (see, for example, Lemma B.1 in [9]). Instead, the main feature of the Wiener randomization (1.10) is that  $(u_0^\omega, u_1^\omega)$  behaves better in terms of integrability. More precisely, if  $u_j \in L^2(\mathbb{R}^d)$ ,  $j = 0, 1$ , then the randomized function  $u_j^\omega$

is almost surely in  $L^p(\mathbb{R}^d)$  for any finite  $p \geq 2$ . See [4]. It is this improved integrability that allows us to construct global solutions to (1.1) below the energy space in a probabilistic manner.

**Remark 1.1.** The uniform decomposition (1.8) comes from the modulation symmetry (of  $L^2(\mathbb{R}^d)$ ), i.e. the translation symmetry on the Fourier side. As such, the uniform decomposition (1.8) and the Wiener randomization (1.10) are closely related to the modulation spaces. See [4] for more discussion on this issue.

**Remark 1.2.** Let  $M = \mathbb{T}^d$ . In this case, if  $\{g_n\}_{n \in \mathbb{Z}^d}$  is a sequence of independent standard Gaussian random variables, then  $\Xi$  in (1.6) represents the white noise on  $\mathbb{T}^d$  and the randomization (1.5) gives the white noise randomization for functions defined on  $\mathbb{T}^d$ . Given  $L > 0$ , let  $\Xi_L$  be the white noise on  $\mathbb{T}_L^d := (\mathbb{R}/L\mathbb{Z})^d \simeq [-\frac{L}{2}, \frac{L}{2})^d$  defined by

$$\Xi_L(x; \omega) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{L^{\frac{d}{2}}} e^{2\pi i \frac{n}{L} \cdot x}.$$

Then, we can also consider the white noise randomization  $u_0^\omega = \Xi_L * u_0$  for functions on  $\mathbb{T}_L^d$ . One of the main features of this randomization is the gain in integrability; if  $u_0 \in L^2(\mathbb{T}_L^d)$ , then the randomized function  $u_0^\omega$  is almost surely in  $L^p(\mathbb{T}_L^d)$  for any finite  $p \geq 2$ . As mentioned above, this improved integrability also holds for the Wiener randomization (1.10) for functions on  $\mathbb{R}^d$ .

Given a function  $u_0$  on  $\mathbb{R}^d$ , one may be tempted to consider an analogous white noise randomization  $u_0^\omega := \Xi_{\mathbb{R}^d} * u_0$  on  $\mathbb{R}^d$ , where  $\Xi_{\mathbb{R}^d}$  is the white noise on  $\mathbb{R}^d$  obtained as the limit<sup>2</sup> of  $\Xi_L$  as  $L \rightarrow \infty$ . Such a randomization, however, is not suitable for our problem due to the lack of (global) integrability. For example, given  $u_0 \in L^2(\mathbb{R}^d)$ , it follows from  $\mathbb{E}[\Xi_{\mathbb{R}^d}(x) \overline{\Xi_{\mathbb{R}^d}(y)}] = \delta(x - y)$  that

$$\begin{aligned} \mathbb{E}[\|\Xi_{\mathbb{R}^d} * u_0\|_{L^2(\mathbb{R}^d)}^2] &= \int \mathbb{E} \left[ \int \Xi_{\mathbb{R}^d}(x - y) u_0(y) dy \overline{\int \Xi_{\mathbb{R}^d}(x - z) u_0(z) dz} \right] dx \\ &= \int \|u_0\|_{L^2(\mathbb{R}^d)}^2 dx = \infty. \end{aligned}$$

This shows that while the white noise randomization is useful in studying evolution equations on  $\mathbb{T}^d$ , it is not suitable on  $\mathbb{R}^d$ , at least for our problem. The Wiener randomization discussed above can be regarded as a suitable adaptation of the white noise randomization on  $\mathbb{R}^d$ , but on a fixed scale. See [5] for the effect of the Wiener randomization based on dilated cubes.

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2. As  $L \rightarrow \infty$ ,  $\Xi_L$  converges in distribution to the white noise  $\Xi_{\mathbb{R}^d}$  on  $C_{\text{loc}}^{0,s}(\mathbb{R}^d; \mathbb{C})$ ,  $s = -\frac{d}{2}-$ , viewed as a Fréchet space endowed with the metric:

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f - g\|_{C^{0,s}([-k,k])}}{1 + \|f - g\|_{C^{0,s}([-k,k])}}.$$

This can be seen from the corresponding convergence (in distribution) of the periodized Brownian motion on  $\mathbb{T}^d$  (represented by the Fourier-Wiener series) to the Brownian motion on  $\mathbb{R}^d$  in  $C_{\text{loc}}^{0,s}(\mathbb{R}^d; \mathbb{C})$  with  $s = 1 - \frac{d}{2}-$ .

**1.3. Main result.** Our main goal in this paper is to prove almost sure global well-posedness of (1.1) on  $\mathbb{R}^3$  below the energy space (Theorem 1.5).

We use the following shorthand notations for products of Sobolev spaces:

$$\mathcal{H}^s(\mathbb{R}^3) := H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \quad \text{and} \quad \dot{\mathcal{H}}^s(\mathbb{R}^3) := \dot{H}^s(\mathbb{R}^3) \times \dot{H}^{s-1}(\mathbb{R}^3).$$

We also denote by  $S(t)$  the propagator for the linear wave equation given by

$$S(t)(u_0, u_1) := \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1. \quad (1.13)$$

We first present the following result on almost sure local well-posedness of (1.1) below the energy space.

**Theorem 1.3** (Almost sure local well-posedness). *Let  $s \in [0, 1)$ . Given  $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^3)$ , let  $(u_0^\omega, u_1^\omega)$  be the Wiener randomization defined in (1.10), satisfying (1.12). Then, the energy-critical defocusing quintic NLW (1.1) on  $\mathbb{R}^3$  is almost surely locally well-posed with respect to the Wiener randomization  $(u_0^\omega, u_1^\omega)$  as initial data. More precisely, there exist  $C, c, \gamma > 0$  such that for each  $T \ll 1$ , there exists a set  $\Omega_T \subset \Omega$  with the following properties:*

- (i)  $P(\Omega_T^c) < C \exp(-\frac{c}{T^\gamma})$ .
- (ii) For each  $\omega \in \Omega_T$ , there exists a unique solution  $u^\omega$  to (1.1) with  $(u^\omega, \partial_t u^\omega)|_{t=0} = (u_0^\omega, u_1^\omega)$  in the class:

$$(S(t)(u_0^\omega, u_1^\omega), \partial_t S(t)(u_0^\omega, u_1^\omega)) + C([-T, T]; \mathcal{H}^1(\mathbb{R}^3)) \subset C([-T, T]; \mathcal{H}^s(\mathbb{R}^3)).$$

Here, uniqueness holds in a ball centered at  $S(\cdot)(u_0^\omega, u_1^\omega)$  in

$$C([-T, T]; \dot{H}^1(\mathbb{R}^3)) \cap L^5([-T, T]; L^{10}(\mathbb{R}^3)).$$

This theorem is in the spirit of the almost sure local well-posedness results in [9, 4, 25]. Namely, given random initial data  $(u_0^\omega, u_1^\omega)$ , denote the linear and nonlinear parts of the solution  $u^\omega$  to (1.1) by

$$z^\omega(t) := S(t)(u_0^\omega, u_1^\omega) \quad \text{and} \quad v^\omega := u^\omega - z^\omega. \quad (1.14)$$

Then, (1.1) can be reformulated as the following perturbed NLW:

$$\begin{cases} \partial_t^2 v^\omega - \Delta v^\omega + (v^\omega + z^\omega)^5 = 0 \\ (v^\omega, \partial_t v^\omega)|_{t=0} = (0, 0). \end{cases} \quad (1.15)$$

In view of the usual deterministic Strichartz estimates (Lemma 2.1) and the probabilistic Strichartz estimates (Lemma 3.2), a simple fixed point argument allows us to construct a solution  $v^\omega$  to (1.15) in  $C([-T, T]; H^1(\mathbb{R}^3))$  for each  $\omega$  belonging to some appropriate set  $\Omega_T$ . This yields Theorem 1.3. As this argument is standard, we omit the proof of Theorem 1.3. See [25] for details.

**Remark 1.4.** (i) Note that the regularity  $s = 0$  is the lowest regularity for which one can prove almost sure local well-posedness by this argument of constructing a solution  $v^\omega$  to (1.15) in  $C([-T, T]; H^1(\mathbb{R}^3))$ . This is due to the fact that the nonlinear Duhamel term in (2.2) gains exactly one derivative.

(ii) In the definition of the Wiener randomization (1.10), we used a smooth cutoff function  $\psi$ . Theorem 1.3 still holds even when we replace  $\psi$  by the sharp characteristic function  $\chi_Q$  of the unit cube  $Q$ . The same comment holds for Theorem 1.5. See also Remark 1.6 (iii) below.

Next, we turn our attention to the global-in-time behavior of solutions with random initial data below the energy space. The following is the main result of this paper.

**Theorem 1.5** (Almost sure global well-posedness). *Let  $s \in (\frac{1}{2}, 1)$ . Given  $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^3)$ , let  $(u_0^\omega, u_1^\omega)$  be the Wiener randomization defined in (1.10), satisfying (1.12). Then, the energy-critical defocusing quintic NLW (1.1) on  $\mathbb{R}^3$  is almost surely globally well-posed with respect to the Wiener randomization  $(u_0^\omega, u_1^\omega)$  as initial data. More precisely, there exists a set  $\Omega_{(u_0, u_1)} \subset \Omega$  of probability 1 such that, for every  $\omega \in \Omega_{(u_0, u_1)}$ , there exists a unique solution  $u$  to (1.1) in the class:*

$$(S(t)(u_0^\omega, u_1^\omega), \partial_t S(t)(u_0^\omega, u_1^\omega)) + C(\mathbb{R}; \mathcal{H}^1(\mathbb{R}^3)) \subset C(\mathbb{R}; \mathcal{H}^s(\mathbb{R}^3)).$$

Before explaining the main ideas of the proof of Theorem 1.5, we first discuss the previous results directly relevant to our problem. Interested readers are referred to [25] for a thorough list of references on almost sure global well-posedness of evolution equations with random initial data.

Previously, Lührmann-Mendelson [23] considered energy-subcritical defocusing NLW on  $\mathbb{R}^3$  with nonlinearity  $|u|^{p-1}u$ ,  $p < 5$ , with random initial data of the form (1.10). In particular, for  $\frac{1}{4}(7 + \sqrt{73}) \simeq 3.89 < p < 5$ , they proved almost sure global well-posedness below the scaling critical Sobolev regularity  $s_{\text{crit}} := \frac{3}{2} - \frac{2}{p-1} < 1$ . Their approach is based on the probabilistic high-low method introduced by Colliander-Oh [13] in the study of the cubic nonlinear Schrödinger equation (NLS) on  $\mathbb{T}$  with random initial data. This method is an adaptation of Bourgain's high-low method [7] to the probabilistic setting and is effective in a subcritical regime. It is, however, not an appropriate tool in our energy-critical setting.

In [25], the second author considered the energy-critical defocusing NLW on  $\mathbb{R}^d$ ,  $d = 4, 5$ , and proved almost sure global well-posedness below the energy space. The main novel approach in [25] is the *probabilistic perturbation theory*. See also Bényi-Oh-Pocovnicu [5]. One of the key ingredients in applying probabilistic perturbation theory was a *probabilistic (a priori) energy bound*. Here, the probabilistic energy bound states that given any  $T, \varepsilon > 0$ , there exists  $\Omega_{T, \varepsilon} \subset \Omega$  with  $P(\Omega_{T, \varepsilon}^c) < \varepsilon$  such that, for all  $\omega \in \Omega_{T, \varepsilon}$ , the solution  $v^\omega$  to the perturbed NLW (1.15) (with the appropriate energy-critical powers for  $d = 4, 5$ ) satisfies

$$\|(v^\omega(t), \partial_t v^\omega(t))\|_{L^\infty([0, T], \dot{H}^1(\mathbb{R}^d))} \leq C(T, \varepsilon) \quad (1.16)$$

for some  $C(T, \varepsilon) > 0$ . Such a probabilistic energy bound was first established by Burq-Tzvetkov [10] in the context of the (energy-subcritical) cubic NLW on  $\mathbb{T}^3$ . In [10, 25], (1.16) was obtained by estimating the growth of the (non-conserved) energy  $E(v^\omega)$  of the solution  $v^\omega$  to (1.15) via probabilistic Strichartz estimates, Sobolev's inequality, and Gronwall's inequality. Such an argument as in [10, 25], however, does not hold for the energy-critical defocusing quintic NLW (1.1) on  $\mathbb{R}^3$ . In particular, the degree of the quintic nonlinearity is too high to close the argument. See Remark 5.1 in [25].

Theorem 1.5 covers the missing case from [23, 25]:  $p = 5$  and  $d = 3$ . This corresponds to the energy-critical NLW in three spatial dimensions, and thus it is of importance from a physical point of view as well as an analytical point of view. As in [25], the main approach to prove Theorem 1.5 is the probabilistic perturbation theory. In the deterministic setting, perturbation theory has played an important role in the study of the energy-critical NLS and NLW [12, 20]. It has also been effective in establishing global well-posedness of NLS with a combined power-type nonlinearity [30, 21]. In our probabilistic approach, we view (1.15) as the defocusing quintic NLW with a (random) perturbation given by  $(v^\omega + z^\omega)^5 - (v^\omega)^5$ . Then, smallness of the perturbation comes from the probabilistic Strichartz estimates (Lemma 3.2) satisfied by the random linear part  $z^\omega$ . In particular, by restricting the analysis to short time intervals, we can make the perturbation small.

In applying perturbation theory in the probabilistic setting in [25], it was essential to have the probabilistic energy bound (1.16). As we pointed out above, however, the approach in [10, 25] does not yield a probabilistic energy bound (1.16) for the perturbed NLW (1.15) on  $\mathbb{R}^3$ . Indeed, this is the main source of difficulty in establishing Theorem 1.5. In order to resolve this issue, we develop a more intricate analysis that will allow us to obtain a suitable replacement of the probabilistic energy bound (1.16). More precisely, we consider a sequence  $\{v_N^\omega\}_{N \geq 1}$ , dyadic of smooth random approximating solutions and establish a *uniform* (in  $N$ ) probabilistic energy bound for  $v_N^\omega$ . See Proposition 4.1 below. The main ingredient in the proof of Proposition 4.1 is a new probabilistic estimate (Proposition 3.3), where we control the  $L_t^\infty$ -norm of random linear solutions. We point out that we only prove a probabilistic energy bound, uniformly in  $N$ , for the approximating random solutions  $v_N^\omega$ . In particular, we do not know how to directly prove a probabilistic energy bound (1.16) for the solution  $v^\omega$  to (1.15). Such a probabilistic energy bound for  $v^\omega$  follows as a corollary to the proof of Theorem 1.5. See Proposition 6.1 below.

Finally, the uniform probabilistic energy bound (Proposition 4.1) combined with the perturbation theory adapted to our setting (Proposition 5.2) yields Theorem 1.5.

We conclude this introduction by stating several remarks.

**Remark 1.6.** (i) The uniqueness statement in Theorem 1.5 holds in the following sense. The set  $\Omega_{(u_0, u_1)}$  in Theorem 1.5 can be written as  $\Omega_{(u_0, u_1)} = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$  with  $P(\Omega_\varepsilon^c) < \varepsilon$ . Given  $\varepsilon > 0$ , for all  $\omega \in \Omega_\varepsilon$  and any finite  $T > 0$ , there exists a sequence of disjoint intervals  $\{I_j\}_{j \in \mathbb{N}}$  covering  $[-T, T]$  such that the solution  $u^\omega$  is unique in some ball centered at  $S(\cdot)(u_0^\omega, u_1^\omega)$  in  $C(I_j, \dot{H}^1(\mathbb{R}^3)) \cap L^5(I_j, L^{10}(\mathbb{R}^3))$  for all  $j \in \mathbb{N}$ . The uniqueness part of Theorem 1.5 is essentially contained in the local-in-time Cauchy theory and we omit its proof. See Theorem 5.3 in [25].

(ii) As in [10, 24, 25], we can enhance the statement in Theorem 1.5 in the following sense. Let  $\mathbf{u}_0 : \Omega \rightarrow \mathcal{H}^s(\mathbb{R}^3)$  be a map given by  $\mathbf{u}_0(\omega) := (u_0^\omega, u_1^\omega)$ , where  $(u_0^\omega, u_1^\omega)$  is as in (1.10). Then, the map  $\mathbf{u}_0$  induces a probability measure  $\mu = \mu_{(u_0, u_1)} = P \circ \mathbf{u}_0^{-1}$  on  $\mathcal{H}^s(\mathbb{R}^3)$ . Arguing as in [25], we can show that there exists a set of  $\mu$ -full measure  $\Sigma \subset \mathcal{H}^s(\mathbb{R}^3)$  such that (a) for any  $(\phi_0, \phi_1) \in \Sigma$ , there exists a unique global solution  $u$  to (1.1) with initial data  $(u, \partial_t u)|_{t=0} = (\phi_0, \phi_1)$  and (b)  $\mu(\Phi(t)(\Sigma)) = 1$  for any  $t \in \mathbb{R}$ , where  $\Phi(t)$  denotes



the solution map of (1.1). Namely, the measure of our initial data set  $\Sigma$  does not become smaller under the dynamics of (1.1).

(iii) As a byproduct of the proof of Theorem 1.5, we obtain the probabilistic energy bound (1.16) for the solution  $v^\omega$  to (1.15). Then, by replacing the smooth cutoff function  $\psi$  with the sharp cutoff function  $\chi_Q$ , we can also obtain the probabilistic continuous dependence of the solution map, and thus probabilistic Hadamard global well-posedness in the sense of [10, 25]. See Remark 1.4 in [25].

(iv) The almost sure global well-posedness result of the energy-critical wave equation on  $\mathbb{R}^d$ ,  $d = 4, 5$ , in [25] holds for  $s > 0$  when  $d = 4$  and  $s \geq 0$  when  $d = 5$ . This is (almost) optimal in view of Remark 1.4 (i). Theorem 1.5 on  $\mathbb{R}^3$ , however, holds only for  $s > \frac{1}{2}$ . This regularity loss appears in establishing a uniform probabilistic energy bound for approximating random solutions (Proposition 4.1). At this point, we do not know how to close this regularity gap.

(v) In view of Theorem 1.5, it is natural to consider the problem of scattering for (1.1) in the probabilistic setting. A key ingredient would be to establish a probabilistic bound on the global space-time Strichartz  $L_t^5 L_x^{10}$ -norm of the solution  $v^\omega$  to (1.15). The probabilistic perturbation theory used for Theorem 1.5, however, only yields a bound on the  $L_t^5 L_x^{10}$ -norm of the solution  $v^\omega$  on short time intervals and does not allow us to establish a global space-time bound. Thus, a new idea is needed to prove probabilistic scattering (for large data).

As in the deterministic setting, there is no such difficulty in the small data case. Indeed, Lührmann-Mendelson [23] proved a probabilistic small data scattering result for (1.1) with large probability. Moreover, even in the large data case, by considering the Wiener randomization on dilated cubes as in [5], one can establish a probabilistic scattering result for (1.1) with large probability. See [5] for details of such an argument. It is worthwhile to note that these results hold only with large probability, i.e. not almost surely.

This paper is organized as follows. In Section 2, we introduce basic notations and recall the deterministic Strichartz estimates. Section 3 covers the necessary probabilistic estimates. In particular, Proposition 3.3 is novel and plays an essential role in proving a uniform probabilistic energy bound for approximating random solutions (Proposition 4.1) in Section 4. In Section 5, we handle the deterministic component of the proof of Theorem 1.5. Then, we present the proof of Theorem 1.5 in Section 6.

In view of the time reversibility of the equation, we only consider positive times in the following.

## 2. NOTATIONS

We say that  $u$  is a solution to the following nonhomogeneous wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u + F = 0 \\ (u, \partial_t u)|_{t=t_0} = (\phi_0, \phi_1) \end{cases} \quad (2.1)$$

on a time interval  $I$  containing  $t_0$ , if  $u$  satisfies the following Duhamel formulation:

$$u(t) = S(t - t_0)(\phi_0, \phi_1) - \int_{t_0}^t \frac{\sin((t - t')|\nabla|)}{|\nabla|} F(t') dt' \quad (2.2)$$

for  $t \in I$ . Here,  $S(\cdot)$  denotes the linear propagator defined in (1.13). We now recall the Strichartz estimates for wave equations on  $\mathbb{R}^3$ . We say that  $(q, r)$  is a  $s$ -wave admissible pair if  $q \geq 2$ ,  $2 \leq r < \infty$ ,

$$\frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, \quad \text{and} \quad \frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s.$$

Then, we have the following Strichartz estimates. See [14, 22, 19] for more discussions on the Strichartz estimates.

**Lemma 2.1.** *Let  $s > 0$ . Let  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  be  $s$ - and  $(1 - s)$ -wave admissible pairs, respectively. Then, we have*

$$\|(u, \partial_t u)\|_{L_t^\infty(I; \dot{\mathcal{H}}_x^s)} + \|u\|_{L_t^q(I; L_x^r)} \lesssim \|(\phi_0, \phi_1)\|_{\dot{\mathcal{H}}^s} + \|F\|_{L_t^{\tilde{q}'}(I; L_x^{\tilde{r}'})} \quad (2.3)$$

for all solutions  $u$  to (2.1) on a time interval  $I \ni t_0$ .

In our argument, we will only use the following wave admissible pairs:  $(5, 10)$  with  $s = 1$  and  $(\infty, 2)$  with  $s = 0$ . For simplicity, we often denote the space  $L_t^q(I; L_x^r)$  by  $L_I^q L_x^r$  or  $L_T^q L_x^r$  if  $I = [0, T]$ .

Next, we briefly go over the Littlewood-Paley theory. Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a smooth bump function supported on  $[-\frac{8}{5}, \frac{8}{5}]$  and  $\varphi \equiv 1$  on  $[-\frac{5}{4}, \frac{5}{4}]$ . Given dyadic  $N \geq 1$ , we set  $\varphi_1(\xi) = \varphi(|\xi|)$  and

$$\varphi_N(\xi) = \varphi\left(\frac{|\xi|}{N}\right) - \varphi\left(\frac{2|\xi|}{N}\right)$$

for  $N \geq 2$ . Then, we define the Littlewood-Paley projection  $\mathbf{P}_N$  as the Fourier multiplier operator with symbol  $\varphi_N$ . Moreover, we define  $\mathbf{P}_{\leq N}$  and  $\mathbf{P}_{\geq N}$  by  $\mathbf{P}_{\leq N} = \sum_{1 \leq M \leq N} \mathbf{P}_M$  and  $\mathbf{P}_{\geq N} = \sum_{M \geq N} \mathbf{P}_M$ .

Lastly, recall Bernstein's inequality:

$$\|\mathbf{P}_{\leq N} f\|_{L^q(\mathbb{R}^3)} \lesssim N^{\frac{3}{p} - \frac{3}{q}} \|\mathbf{P}_{\leq N} f\|_{L^p(\mathbb{R}^3)}, \quad 1 \leq p \leq q \leq \infty. \quad (2.4)$$

As an immediate corollary of (2.4), we have, for all  $n \in \mathbb{Z}^3$ ,

$$\|\psi(D - n)\phi\|_{L^q(\mathbb{R}^3)} \lesssim \|\psi(D - n)\phi\|_{L^p(\mathbb{R}^3)}, \quad 1 \leq p \leq q \leq \infty. \quad (2.5)$$

### 3. PROBABILISTIC ESTIMATES

In this section, we first review some basic properties of randomized functions. Then, we present the main new probabilistic estimate (Proposition 3.3), controlling the  $L_t^\infty$ -norm of random linear solutions.

First recall the following probabilistic estimate. See [9] for the proof.

**Lemma 3.1.** *Let  $\{g_n\}_{n \in \mathbb{Z}^3}$  be a sequence of mean zero complex-valued, random variables such that  $g_{-n} = \overline{g_n}$  for all  $n \in \mathbb{Z}^3$ . With  $\mathcal{I}$  as in (1.9), assume that  $g_0$ ,  $\text{Re } g_n$ , and  $\text{Im } g_n$ ,*

$n \in \mathcal{I}$ , are independent. Moreover, assume that (1.12) is satisfied. Then, there exists  $C > 0$  such that the following holds:

$$\left\| \sum_{n \in \mathbb{Z}^3} g_n(\omega) c_n \right\|_{L^p(\Omega)} \leq C \sqrt{p} \|c_n\|_{\ell_n^2(\mathbb{Z}^3)}$$

for any  $p \geq 2$  and any sequence  $\{c_n\} \in \ell^2(\mathbb{Z}^3)$  satisfying  $c_{-n} = \overline{c_n}$  for all  $n \in \mathbb{Z}^3$ .

Next, we recall the local-in-time probabilistic Strichartz estimates.

**Lemma 3.2** (Proposition 2.3 in [25]). *Given a pair  $(u_0, u_1)$  of real-valued functions defined on  $\mathbb{R}^3$ , let  $(u_0^\omega, u_1^\omega)$  be the Wiener randomization defined in (1.10), satisfying (1.12). Let  $I = [a, b] \subset \mathbb{R}$  be a compact time interval.*

(i) *If  $(u_0, u_1) \in \dot{\mathcal{H}}^0(\mathbb{R}^3)$ , then given  $1 \leq q < \infty$  and  $2 \leq r < \infty$ , there exist  $C, c > 0$  such that*

$$P\left(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)} > \lambda\right) \leq C \exp\left(-c \frac{\lambda^2}{|I|^{\frac{2}{q}} \|(u_0, u_1)\|_{\dot{\mathcal{H}}^0}^2}\right).$$

(ii) *If  $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^3)$ , then given  $1 \leq q < \infty$ ,  $2 \leq r \leq \infty$ , there exist  $C, c > 0$  such that*

$$P\left(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^q(I; L_x^r)} > \lambda\right) \leq C \exp\left(-c \frac{\lambda^2}{\max(1, |a|^2, |b|^2) |I|^{\frac{2}{q}} \|(u_0, u_1)\|_{\mathcal{H}^s}^2}\right)$$

for (ii.a)  $s = 0$  if  $r < \infty$  and (ii.b)  $s > 0$  if  $r = \infty$ .

Lemma 3.2 plays an essential role in the proof of Theorem 1.3. The proof of Lemma 3.2 follows from Lemma 3.1 and (2.5). See [23, 25] for details.

The following proposition allows us to obtain a probabilistic estimate involving the  $L_t^\infty$ -norm and plays an important role in establishing a probabilistic energy bound. See Proposition 4.1 below.

Define  $\tilde{S}(t)$  by

$$\tilde{S}(t)(u_0, u_1) := -\frac{|\nabla|}{\langle \nabla \rangle} \sin(t|\nabla|)u_0 + \frac{\cos(t|\nabla|)}{\langle \nabla \rangle} u_1. \quad (3.1)$$

Namely, we have  $\partial_t S(t)(u_0, u_1) = \langle \nabla \rangle \tilde{S}(t)(u_0, u_1)$ .

**Proposition 3.3.** *Given a pair  $(u_0, u_1)$  of real-valued functions defined on  $\mathbb{R}^3$ , let  $(u_0^\omega, u_1^\omega)$  be the Wiener randomization defined in (1.10), satisfying (1.12). Let  $T > 0$  and  $S^*(t) = S(t)$  or  $\tilde{S}(t)$  defined in (1.13) and (3.1), respectively. Then, for  $2 \leq r \leq \infty$ , we have*

$$\begin{aligned} P\left(\|S^*(t)(u_0^\omega, u_1^\omega)\|_{L_t^\infty([0, T]; L_x^r(\mathbb{R}^3))} > \lambda\right) \\ \leq C(1 + T) \exp\left(-c \frac{\lambda^2}{\max(1, T^2) \|(u_0, u_1)\|_{\mathcal{H}^\varepsilon(\mathbb{R}^3)}^2}\right) \end{aligned} \quad (3.2)$$

for any  $\varepsilon > 0$ , where the constants  $C$  and  $c$  depend only on  $r$  and  $\varepsilon$ .

Proposition 3.3 follows as a corollary to the following lemma. Let  $S_+(t)$  and  $S_-(t)$  be the linear propagators for the half wave equations defined by

$$S_{\pm}(t)\phi := \mathcal{F}^{-1}(e^{\pm i|\xi|t}\widehat{\phi}(\xi)).$$

Given  $\phi \in H^s(\mathbb{R}^3)$ , we define its randomization  $\phi^\omega$  by

$$\phi^\omega := \sum_{n \in \mathbb{Z}^d} g_{n,0}(\omega) \psi(D - n) \phi$$

as in the first component of (1.10). Then, we have the following tail estimate on the size of  $S_{\pm}(t)\phi^\omega$  over a time interval of length 1.

**Lemma 3.4.** *Let  $j \in \mathbb{N} \cup \{0\}$  and  $2 \leq r \leq \infty$ . Given any  $\varepsilon > 0$ , there exist constants  $C, c > 0$ , depending only on  $r$  and  $\varepsilon$ , such that*

$$P\left(\|S_{\pm}(t)\phi^\omega\|_{L_t^\infty([j,j+1];L_x^r(\mathbb{R}^3))} > \lambda\right) \leq C \exp\left(-c \frac{\lambda^2}{\|\phi\|_{H^\varepsilon(\mathbb{R}^3)}^2}\right), \quad (3.3)$$

$$P\left(\left\|\frac{\sin(t|\nabla|)}{|\nabla|}\phi^\omega\right\|_{L_t^\infty([j,j+1];L_x^r(\mathbb{R}^3))} > \lambda\right) \leq C \exp\left(-c \frac{\lambda^2}{\max(1,j^2)\|\phi\|_{H^{\varepsilon-1}(\mathbb{R}^3)}^2}\right). \quad (3.4)$$

Assuming Lemma 3.4, we first present the proof of Proposition 3.3.

*Proof of Proposition 3.3.* We only consider the case  $S^*(t) = S(t)$  and  $T \geq 1$ . When  $S^*(t) = \tilde{S}(t)$ , (3.2) holds without the factor  $T^2$  in the exponent. By subadditivity and Lemma 3.4, we have

$$\begin{aligned} P\left(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^\infty([0,T];L_x^r(\mathbb{R}^3))} > \lambda\right) &\leq P\left(\max_{j=0,\dots,[T]} \|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^\infty([j,j+1];L_x^r(\mathbb{R}^3))} > \lambda\right) \\ &\leq \sum_{j=0}^{[T]} P\left(\|S(t)(u_0^\omega, u_1^\omega)\|_{L_t^\infty([j,j+1];L_x^r(\mathbb{R}^3))} > \lambda\right) \\ &\leq \sum_{j=0}^{[T]} P\left(\|\cos(t|\nabla|)u_0^\omega\|_{L_t^\infty([j,j+1];L_x^r(\mathbb{R}^3))} > \frac{\lambda}{2}\right) \\ &\quad + \sum_{j=0}^{[T]} P\left(\left\|\frac{\sin(t|\nabla|)}{|\nabla|}u_1^\omega\right\|_{L_t^\infty([j,j+1];L_x^r(\mathbb{R}^3))} > \frac{\lambda}{2}\right) \\ &\leq C([T] + 1) \exp\left(-c \frac{\lambda^2}{T^2\|(u_0, u_1)\|_{\mathcal{H}^\varepsilon(\mathbb{R}^3)}^2}\right). \end{aligned}$$

Here,  $[T]$  denotes the integer part of  $T$ . □

Finally, we prove Lemma 3.4.

*Proof of Lemma 3.4.* We first prove (3.3). Define  $z_{\pm}^\omega(t)$  and  $z_{\pm}(t)$  by

$$z_{\pm}^\omega(t) := S_{\pm}(t)\phi^\omega \quad \text{and} \quad z_{\pm}(t) := S_{\pm}(t)\phi.$$

**Part 1 (a):** We first consider the case  $r < \infty$ . Without loss of generality, assume  $j = 0$ . For  $k \in \mathbb{N} \cup \{0\}$ , let  $\{t_{\ell,k} : \ell = 0, 1, \dots, 2^k\}$  be  $2^k + 1$  equally spaced points on  $[0, 1]$ , i.e.  $t_{0,k} = 0$  and  $t_{\ell,k} - t_{\ell-1,k} = 2^{-k}$  for  $\ell = 1, \dots, 2^k$ . Then, given  $t \in [0, 1]$ , we have

$$z_{\pm}^{\omega}(t) = \sum_{k=1}^{\infty} (z_{\pm}^{\omega}(t_{\ell_k,k}) - z_{\pm}^{\omega}(t_{\ell_{k-1},k-1})) + z_{\pm}^{\omega}(0) \quad (3.5)$$

for some  $\ell_k = \ell_k(t) \in \{0, \dots, 2^k\}$ .<sup>3</sup>

We consider the  $L_x^r$ -norm with the (inhomogeneous) Littlewood-Paley decomposition. Then, by the square function estimate and Minkowski's integral inequality, we have

$$\|z_{\pm}^{\omega}(t)\|_{L_x^r} \sim \left\| \left( \sum_{\substack{N \geq 1 \\ \text{dyadic}}} |\mathbf{P}_N z_{\pm}^{\omega}(t)|^2 \right)^{\frac{1}{2}} \right\|_{L_x^r} \leq \left( \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \|\mathbf{P}_N z_{\pm}^{\omega}(t)\|_{L_x^r}^2 \right)^{\frac{1}{2}}. \quad (3.6)$$

Then, from (3.5) and (3.6), we have

$$\|z_{\pm}^{\omega}\|_{L_t^{\infty}([0,1]; L_x^r)} \lesssim \left( \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \left( \sum_{k=1}^{\infty} \max_{0 \leq \ell_k \leq 2^k} \|\mathbf{P}_N(z_{\pm}^{\omega}(t_{\ell_k,k}) - z_{\pm}^{\omega}(t_{\ell_{k-1},k-1}))\|_{L_x^r} \right)^2 \right)^{\frac{1}{2}} + \|z_{\pm}^{\omega}(0)\|_{L_x^r},$$

where  $t_{\ell_{k-1},k-1}$  is one of the  $2^{(k-1)} + 1$  equally spaced points such that

$$|t_{\ell_k,k} - t_{\ell_{k-1},k-1}| \leq 2^{-k}. \quad (3.7)$$

Hence, for  $p \geq 2$ , we have

$$\begin{aligned} & \left( \mathbb{E} [\|z_{\pm}^{\omega}\|_{L_t^{\infty}([0,1]; L_x^r)}]^p \right)^{\frac{1}{p}} \\ & \lesssim \left( \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \left( \sum_{k=1}^{\infty} \left( \mathbb{E} \left[ \max_{0 \leq \ell_k \leq 2^k} \|\mathbf{P}_N(z_{\pm}^{\omega}(t_{\ell_k,k}) - z_{\pm}^{\omega}(t_{\ell_{k-1},k-1}))\|_{L_x^r} \right]^p \right)^{\frac{1}{p}} \right)^2 \right)^{\frac{1}{2}} \\ & + \left( \mathbb{E} [\|z_{\pm}^{\omega}(0)\|_{L_x^r}]^p \right)^{\frac{1}{p}}. \end{aligned} \quad (3.8)$$

Note that it follows from Lemma 3.1 and (2.5) that the second term on the right-hand side of (3.8) can be bounded by

$$\left( \mathbb{E} [\|z_{\pm}^{\omega}(0)\|_{L_x^r}]^p \right)^{\frac{1}{p}} \lesssim \sqrt{p} \|\phi\|_{L_x^2} \quad (3.9)$$

for  $p \geq r$ .

In the following, we first estimate

$$I_N := \sum_{k=1}^{\infty} \left( \mathbb{E} \left[ \max_{0 \leq \ell_k \leq 2^k} \|\mathbf{P}_N(z_{\pm}^{\omega}(t_{\ell_k,k}) - z_{\pm}^{\omega}(t_{\ell_{k-1},k-1}))\|_{L_x^r} \right]^p \right)^{\frac{1}{p}}$$

for each dyadic  $N \geq 1$ . Let

$$q_k := \max(\log 2^k, p, r) \sim \log 2^k + p. \quad (3.10)$$

---

3. Think of the binary expansion of this given  $t \in [0, 1]$ . Then,  $t_{\ell_k,k}$  can be given by the partial sum of this binary expansion up to order  $k$ .

Then, we have

$$I_N \leq \sum_{k=1}^{\infty} \left( \sum_{\ell_k=0}^{2^k} \mathbb{E} \left\| \mathbf{P}_N(z_{\pm}^{\omega}(t_{\ell_k,k}) - z_{\pm}^{\omega}(t_{\ell'_{k-1},k-1})) \right\|_{L_x^r}^{q_k} \right)^{\frac{1}{q_k}}$$

Noting that  $(2^k + 1)^{\frac{1}{q_k}} \lesssim 1$  and applying Lemma 3.1,

$$\begin{aligned} &\lesssim \sum_{k=1}^{\infty} \max_{0 \leq \ell_k \leq 2^k} \left( \mathbb{E} \left\| \mathbf{P}_N(z_{\pm}^{\omega}(t_{\ell_k,k}) - z_{\pm}^{\omega}(t_{\ell'_{k-1},k-1})) \right\|_{L_x^r}^{q_k} \right)^{\frac{1}{q_k}} \\ &\lesssim \sum_{k=1}^{\infty} \sqrt{q_k} \max_{0 \leq \ell_k \leq 2^k} \left\| \left\| \mathbf{P}_N \psi(D-n)(z_{\pm}(t_{\ell_k,k}) - z_{\pm}(t_{\ell'_{k-1},k-1})) \right\|_{L_x^r} \right\|_{\ell_{|n| \sim N}^2}^2 \end{aligned}$$

By (2.5), we have

$$\lesssim \sum_{k=1}^{\infty} \sqrt{q_k} \max_{0 \leq \ell_k \leq 2^k} \left\| \left\| \mathbf{P}_N \psi(D-n)(z_{\pm}(t_{\ell_k,k}) - z_{\pm}(t_{\ell'_{k-1},k-1})) \right\|_{L_x^2} \right\|_{\ell_{|n| \sim N}^2}^2. \quad (3.11)$$

Then, by (3.7) we have

$$\begin{aligned} &\left\| \mathbf{P}_N \psi(D-n)(z_{\pm}(t_{\ell_k,k}) - z_{\pm}(t_{\ell'_{k-1},k-1})) \right\|_{L_x^2} \\ &= \left( \int_{|\xi| \sim N} \left| e^{\pm i|\xi|t_{\ell_k,k}} - e^{\pm i|\xi|t_{\ell'_{k-1},k-1}} \right|^2 |\psi(\xi - n)\widehat{\phi}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\lesssim \min(1, 2^{-k}N) \left\| \mathbf{P}_N \psi(D-n)\phi \right\|_{L_x^2}. \end{aligned} \quad (3.12)$$

Hence, from (3.11) and (3.12), it follows that

$$I_N \lesssim \sum_{k=1}^{\infty} \sqrt{q_k} \min(1, 2^{-k}N) \left\| \mathbf{P}_N \phi \right\|_{L_x^2}. \quad (3.13)$$

Now, we separate the summation into  $2^{-k}N \geq 1$  and  $2^{-k}N < 1$  and estimate the contribution from each case. Note that from (3.10), we have

$$\sqrt{q_k} \lesssim \sqrt{\log 2^k} \cdot \sqrt{p}. \quad (3.14)$$

• **Case 1:**  $2^{-k}N \geq 1$ .

In this case, from (3.14), we have

$$\sqrt{q_k} \lesssim \sqrt{\log N} \cdot \sqrt{p}$$

Hence, we have

$$(3.13) \leq C_r (\log N)^{\frac{3}{2}} \sqrt{p} \left\| \mathbf{P}_N \phi \right\|_{L^2} \leq C_{r,\varepsilon} \sqrt{p} \left\| \mathbf{P}_N \phi \right\|_{H^\varepsilon} \quad (3.15)$$

for any  $\varepsilon > 0$ .

• **Case 2:**  $2^{-k}N < 1$ .

From (3.14), we have

$$\begin{aligned} (3.13) &\leq C_r \sqrt{p} \sum_{k \gtrsim \log N}^{\infty} (\log 2^k)^{\frac{1}{2}} 2^{-k} N \|\mathbf{P}_N \phi\|_{L^2} \lesssim C_r \sqrt{p} (\log N)^{\frac{1}{2}} \|\mathbf{P}_N \phi\|_{L^2} \\ &\leq C_{r,\varepsilon} \sqrt{p} \|\mathbf{P}_N \phi\|_{H^\varepsilon} \end{aligned} \quad (3.16)$$

for any  $\varepsilon > 0$ .

Finally, putting (3.8), (3.9), (3.13), (3.15), and (3.16) together, we obtain

$$\left( \mathbb{E} [\|z^\omega\|_{L_t^\infty([0,1];L_x^r)}]^p \right)^{\frac{1}{p}} \leq C_{r,\varepsilon} \sqrt{p} \|\phi\|_{H_x^\varepsilon}$$

for all  $p \geq r$  and  $\varepsilon > 0$ . The rest follows from a standard argument using Chebyshev's inequality.

**Part 1 (b):** Next, we consider the case  $r = \infty$ . Then, it follows from Sobolev embedding that, given any  $\varepsilon > 0$ , there exists large  $\tilde{r} \gg 1$  with  $\varepsilon \tilde{r} > 3$  such that

$$P\left(\|S_\pm(t)\phi^\omega\|_{L_t^\infty([j,j+1];L_x^\infty(\mathbb{R}^3))} > \lambda\right) \leq P\left(\|\langle \nabla \rangle^\varepsilon S_\pm(t)\phi^\omega\|_{L_t^\infty([j,j+1];L_x^{\tilde{r}}(\mathbb{R}^3))} > C\lambda\right).$$

Then, the rest follows from the argument in Part 1 (a).

**Part 2:** Next, we briefly discuss how to prove (3.4) when  $r < \infty$ . Letting

$$Z^\omega(t) := \frac{\sin(t|\nabla|)}{|\nabla|} \phi^\omega \quad \text{and} \quad Z(t) := \frac{\sin(t|\nabla|)}{|\nabla|} \phi$$

and repeating the argument in Part 1 (but on  $[j, j+1]$  instead of  $[0, 1]$ ), we have

$$\begin{aligned} &\left( \mathbb{E} [\|Z^\omega\|_{L_t^\infty([j,j+1];L_x^r)}]^p \right)^{\frac{1}{p}} \\ &\lesssim \left( \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \left( \sum_{k=1}^{\infty} \left( \mathbb{E} \left[ \max_{0 \leq \ell_k \leq 2^k} \|\mathbf{P}_N(Z^\omega(t_{\ell_k,k}) - Z^\omega(t_{\ell'_{k-1},k-1}))\|_{L_x^r} \right]^p \right)^{\frac{1}{p}} \right)^2 \right)^{\frac{1}{2}} \\ &\quad + \left( \mathbb{E} [\|Z^\omega(j)\|_{L_x^r}]^p \right)^{\frac{1}{p}} =: \text{I} + \text{II}. \end{aligned}$$

When  $j = 0$ , then we have  $\text{II} = 0$ . When  $j \geq 1$ , we argue as in the proof of Proposition 2.3 (ii) in [25] and obtain

$$\text{II} \lesssim \sqrt{p} \max(1, j) \|\phi\|_{H^{-1}} \quad (3.17)$$

for  $p \geq r$ . As for I, we simply repeat the computations in Part 1 with a modification in (3.12):

$$\begin{aligned} &\|\mathbf{P}_N \psi(D-n)(Z(t_{\ell_k,k}) - Z(t_{\ell'_{k-1},k-1}))\|_{L_x^2} \\ &\sim \left( \int_{|\xi| \sim N} \left| \frac{e^{\pm i|\xi|t_{\ell_k,k}} - e^{\pm i|\xi|t_{\ell'_{k-1},k-1}}}{\xi} \right|^2 |\psi(\xi-n)\widehat{\phi}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\lesssim \min(N^{-1}, 2^{-k}) \|\mathbf{P}_N \psi(D-n)\phi\|_{L_x^2}. \end{aligned}$$

This modification yields

$$\text{I} \lesssim C_{r,\varepsilon} \sqrt{p} \|\phi\|_{H^{\varepsilon-1}}. \quad (3.18)$$

Then, the desired estimate (3.4) follows from (3.17) and (3.18).  $\square$

#### 4. UNIFORM PROBABILISTIC ENERGY BOUND FOR APPROXIMATING SOLUTIONS

Let  $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^3)$  with  $\frac{1}{2} < s < 1$ . Given  $N \geq 1$  dyadic, define  $u_{j,N}^\omega$ ,  $j = 0, 1$ , by

$$u_{j,N}^\omega := \mathbf{P}_{\leq N} u_j^\omega = \sum_{n \in \mathbb{Z}^3} g_{n,j}(\omega) \mathbf{P}_{\leq N} \psi(D - n) u_j. \quad (4.1)$$

Note that we have  $(u_{0,N}^\omega, u_{1,N}^\omega) \in \mathcal{H}^\infty(\mathbb{R}^3)$ . Let  $u_N$  be the smooth global-in-time solution to (1.1) with initial data

$$(u_N, \partial_t u_N)|_{t=0} = (u_{0,N}^\omega, u_{1,N}^\omega),$$

and denote by  $z_N = z_N^\omega$  and  $v_N = v_N^\omega$  the linear and nonlinear parts of  $u_N$ . Namely,

$$z_N(t) := S(t)(u_{0,N}^\omega, u_{1,N}^\omega) \quad \text{and} \quad v_N := u_N - z_N. \quad (4.2)$$

In particular,  $v_N$  is the smooth global solution to the following perturbed NLW:

$$\begin{cases} \partial_t^2 v_N - \Delta v_N + (v_N + z_N)^5 = 0, \\ (v_N, \partial_t v_N)|_{t=0} = (0, 0). \end{cases} \quad (4.3)$$

It follows from the conservation of the energy of  $u_N$  and the unitarity of the linear propagator that we have  $\|(v_N^\omega, \partial_t v_N^\omega)\|_{L^\infty(\mathbb{R}; \dot{\mathcal{H}}^1(\mathbb{R}^3))} \leq C(N, \omega) < \infty$  for each  $N \in \mathbb{N}$ . There is, however, no uniform control on the size of  $v_N$ , independent of  $N$ , since the  $\dot{H}^1$ -norm of  $z_N$  tends to infinity almost surely as  $N \rightarrow \infty$ .

The following proposition establishes a probabilistic energy bound on  $v_N$ , *independent* of dyadic  $N \geq 1$ , and plays an important role in the proof of Theorem 1.5.

**Proposition 4.1.** *Let  $s \in (\frac{1}{2}, 1)$  and  $N \geq 1$  dyadic. Given  $T, \varepsilon > 0$ , there exists  $\tilde{\Omega}_{N,T,\varepsilon} \subset \Omega$  such that*

- (i)  $P(\tilde{\Omega}_{N,T,\varepsilon}^c) < \varepsilon$ ,
- (ii) *There exists a finite constant  $C(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}) > 0$  such that the following energy bound holds:*

$$\sup_{t \in [0, T]} \|(v_N^\omega(t), \partial_t v_N^\omega(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} \leq C(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}), \quad (4.4)$$

*for all solutions  $v_N^\omega$  to (4.3) with  $\omega \in \tilde{\Omega}_{N,T,\varepsilon}$ .*

*Note that the constant  $C(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)})$  is independent of dyadic  $N \geq 1$ .*

*Proof.* First, note that it suffices to prove

$$\sup_{t \in [0, T]} \|(v_N^\omega(t), \partial_t v_N^\omega(t))\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)} \leq C(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}). \quad (4.5)$$

Indeed, (4.4) follows from (4.5) and

$$\|v_N^\omega(t)\|_{L_x^2(\mathbb{R}^3)} = \left\| \int_0^t \partial_t v_N^\omega(t') dt' \right\|_{L_x^2} \leq T \|\partial_t v_N^\omega\|_{L_T^\infty L_x^2} \leq C(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}). \quad (4.6)$$



Let  $z_N(t)$  be as in (4.2) and  $\tilde{z}(t) = \tilde{z}_N^\omega(t) := \tilde{S}(t)(u_{0,N}^\omega, u_{1,N}^\omega)$  with  $\tilde{S}(t)$  defined in (3.1). Let  $\delta > 0$  sufficiently small such that  $\frac{1}{2} + \delta < s$ . For fixed  $T, \varepsilon > 0$ , we define  $\tilde{\Omega}_{N,T,\varepsilon}$  by

$$\tilde{\Omega}_{N,T,\varepsilon} = \{\omega : \|z_N^\omega\|_{L_{T,x}^{10}}^{10} + \|z_N^\omega\|_{L_T^\infty L_x^6}^6 + \|z_N^\omega\|_{L_{T,x}^\infty}^2 + \|\tilde{z}_N^\omega\|_{L_{T,x}^6}^6 + \|\langle \nabla \rangle^{s-\delta} \tilde{z}_N^\omega\|_{L_{T,x}^\infty} \leq \lambda\},$$

where  $\lambda = \lambda(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}) > 0$  is chosen such that  $P(\tilde{\Omega}_{N,T,\varepsilon}^c) < \varepsilon$ . Note that the existence of such  $\lambda(T, \varepsilon)$  is guaranteed by Lemma 3.2 and Proposition 3.3. Moreover,  $\lambda(T, \varepsilon)$  can be chosen to be independent of  $N$ .

In the following, we prove

$$\sup_{t \in [0, T]} E(v_N^\omega(t)) \leq C(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}) \quad (4.7)$$

for  $\omega \in \tilde{\Omega}_{N,T,\varepsilon}$ . Then, (4.5) follows from the coercivity of the energy  $E$ .

For simplicity, we denote  $v_N^\omega$  and  $z_N^\omega$  by  $v$  and  $z$ , in the following. By differentiating  $E(v)$  in time, we have

$$\begin{aligned} \frac{d}{dt} E(v)(t) &= \int_{\mathbb{R}^3} \partial_t v (\partial_t^2 v - \Delta v + v^5) dx = - \int_{\mathbb{R}^3} \partial_t v ((z+v)^5 - v^5) dx \\ &= - \int_{\mathbb{R}^3} \partial_t v (5zv^4 + \mathcal{N}(z, v)) dx, \end{aligned}$$

where

$$\mathcal{N}(z, v) := 10z^2v^3 + 10z^3v^2 + 5z^4v + z^5.$$

By integrating in time, we have

$$\begin{aligned} E(v)(t) &= \underbrace{E(v)(0)}_{=0} - \int_0^t \int_{\mathbb{R}^3} \partial_t v(t') [5z(t')v(t')^4 + \mathcal{N}(z, v)(t')] dx dt' \\ &= - \int_{\mathbb{R}^3} \int_0^t z(t') \partial_t (v(t')^5) dt' dx - \int_0^t \int_{\mathbb{R}^3} \partial_t v(t') \mathcal{N}(z, v)(t') dx dt' \\ &=: \text{I}(t) + \text{II}(t), \end{aligned} \quad (4.8)$$

for  $t \in [0, T]$ . Noting that

$$|\mathcal{N}(z, v)(t')| \lesssim |z(t')^2 v(t')^3| + |z(t')|^5,$$

we have

$$\begin{aligned} |\text{II}(t)| &\lesssim \int_0^t \|\partial_t v(t')\|_{L_x^2} \|z(t')\|_{L_x^\infty}^2 \|v(t')\|_{L_x^6}^3 dt' + \int_0^t \|\partial_t v(t')\|_{L_x^2} \|z(t')\|_{L_x^{10}}^5 dt' \\ &\lesssim (1 + \|z\|_{L_{T,x}^\infty}^2) \int_0^t E(v)(t') dt' + \|z\|_{L_{T,x}^{10}}^{10}. \end{aligned} \quad (4.9)$$

Next, we control the term  $\text{I}(t)$ . Note that  $v(0) \equiv 0$  and  $v = v_N^\omega$  is smooth, both in  $x$  and  $t$ . Then, by integration by parts in time, we have

$$\text{I}(t) = - \int_{\mathbb{R}^3} z(t) v(t)^5 dx + \int_{\mathbb{R}^3} \int_0^t \partial_t z(t') v(t')^5 dt' dx =: \text{I}_1(t) + \text{I}_2(t). \quad (4.10)$$

As for the first term  $\text{I}_1(t)$ , we bound it by

$$|\text{I}_1(t)| \lesssim a^{-6} \|z(t)\|_{L_x^6}^6 + a^{\frac{6}{5}} \|v(t)\|_{L_x^6}^6 \lesssim a^{-6} \|z\|_{L_T^\infty L_x^6}^6 + a^{\frac{6}{5}} E(v)(t). \quad (4.11)$$

for some small constant  $a > 0$  (to be chosen later).

It remains to estimate the second term  $I_2(t)$  in (4.10). Noting that  $z(t)$  solves the linear wave equation, we have

$$I_2(t) = \int_{\mathbb{R}^3} \int_0^t \partial_t z(t') v(t')^5 dt' dx = \int_0^t \int_{\mathbb{R}^3} \langle \nabla \rangle \tilde{z}(t') \cdot v(t')^5 dx dt'. \quad (4.12)$$

Define  $\mathcal{I}(t)$  by

$$\mathcal{I}(t) := \int_{\mathbb{R}^3} \langle \nabla \rangle \tilde{z}(t) \cdot v(t)^5 dx \sim \sum_{k=-1}^1 \sum_{\substack{M \geq 1 \\ \text{dyadic}}} M \int_{\mathbb{R}^3} \mathbf{P}_{2^k M} \tilde{z}(t) \mathbf{P}_M [v(t)^5] dx,$$

with the understanding that  $\mathbf{P}_{2^{-1}} = 0$ .

• **Case 1:**  $M = 1$ .

By Young's and Bernstein's inequalities, we have

$$|I_2(t)| \lesssim \|\tilde{z}\|_{L_{T,x}^6}^6 + \int_0^t \|v(t')\|_{L_x^6}^6 dt' \lesssim \|\tilde{z}\|_{L_{T,x}^6}^6 + \int_0^t E(v)(t') dt'. \quad (4.13)$$

• **Case 2:**  $M \geq 2$ .

We write

$$v^5 = \sum_{\substack{M_j \geq 1 \\ \text{dyadic}}} \prod_{j=1}^5 \mathbf{P}_{M_j} v,$$

and assume that  $M_1 \geq M_2 \geq \dots \geq M_5$  without loss of generality. Note that we have  $\mathbf{P}_M[v(t)^5] = 0$  unless  $M_1 \gtrsim M$ . With  $M_1 \gtrsim M$  and using Hölder's inequality, we have

$$|\mathcal{I}(t)| \lesssim \sum_{k=-1}^1 \sum_{\substack{M \geq 2 \\ \text{dyadic}}} \sum_{\substack{M_1, \dots, M_5 \\ M_1 \gtrsim M}} \|\langle \nabla \rangle^{s-\delta} \mathbf{P}_{2^k M} \tilde{z}(t)\|_{L_x^\infty} M_1^{1-s+\delta} \left\| \prod_{j=1}^5 \mathbf{P}_{M_j} v(t) \right\|_{L_x^1}$$

Summing over dyadic  $M, M_1, \dots, M_5$  (with a slight loss in a power of  $M_1$ ) and applying Bernstein's inequality followed by Young's inequality,

$$\begin{aligned} &\lesssim \sup_{M_1, \dots, M_5} \|\langle \nabla \rangle^{s-\delta} \tilde{z}(t)\|_{L_x^\infty} M_1^{1-s+\delta} \left\| \mathbf{P}_{M_1} v(t) \prod_{j=2}^5 \mathbf{P}_{M_j} v(t) \right\|_{L_x^1} \\ &\lesssim \sup_{M_1, \dots, M_5} \|\langle \nabla \rangle^{s-\delta} \tilde{z}(t)\|_{L_x^\infty} \left\{ \|M_1^{1-s+\delta} \mathbf{P}_{M_1} v(t)\|_{L_x^3}^3 + \left\| \prod_{j=2}^5 \mathbf{P}_{M_j} v(t) \right\|_{L_x^{\frac{3}{2}}}^{\frac{3}{2}} \right\} \\ &\lesssim \sup_{M_1, \dots, M_5} \|\langle \nabla \rangle^{s-\delta} \tilde{z}(t)\|_{L_x^\infty} \left\{ M_1^{3(1-s+\delta)} \|\mathbf{P}_{M_1} v(t)\|_{L_x^3}^3 + \prod_{j=2}^5 \|\mathbf{P}_{M_j} v(t)\|_{L_x^6}^{\frac{3}{2}} \right\} \\ &\lesssim \sup_{M_1} \|\langle \nabla \rangle^{s-\delta} \tilde{z}(t)\|_{L_x^\infty} \left\{ M_1^{3(1-s+\delta)} \|\mathbf{P}_{M_1} v(t)\|_{L_x^3}^3 + \|v(t)\|_{L_x^6}^6 \right\} \end{aligned}$$

By interpolating  $L^3$  between  $L^2$  and  $L^6$  and then applying Young's inequality,

$$\begin{aligned} &\lesssim \sup_{M_1} \|\langle \nabla \rangle^{s-\delta} \tilde{z}(t)\|_{L_x^\infty} \left\{ \|M_1^{2(1-s+\delta+)} \mathbf{P}_{M_1} v\|_{L_x^2}^{\frac{3}{2}} \|\mathbf{P}_{M_1} v\|_{L_x^6}^{\frac{3}{2}} + E(v) \right\} \\ &\lesssim \sup_{M_1} \|\langle \nabla \rangle^{s-\delta} \tilde{z}(t)\|_{L_x^\infty} \left\{ \|M_1^{2(1-s+\delta+)} \mathbf{P}_{M_1} v\|_{L_x^2}^2 + \|\mathbf{P}_{M_1} v\|_{L_x^6}^6 + E(v) \right\} \\ &\lesssim \|\langle \nabla \rangle^{s-\delta} \tilde{z}(t)\|_{L_x^\infty} E(v), \end{aligned} \quad (4.14)$$

where the last inequality follows from Bernstein's inequality as long as  $2(1-s+\delta+) \leq 1$ , i.e.  $s > \frac{1}{2} + \delta$ .

Hence, from (4.12), (4.13), and (4.14), we obtain

$$|I_2(t)| \lesssim \|\tilde{z}\|_{L_{T,x}^6}^6 + \left( 1 + \|\langle \nabla \rangle^{s-\delta} \tilde{z}\|_{L_{T,x}^\infty} \right) \int_0^t E(v)(t') dt'. \quad (4.15)$$

By choosing sufficiently small  $a > 0$ , it follows from (4.8), (4.9), (4.10), (4.11), and (4.15) that

$$E(v)(t) \leq C_1(z, \tilde{z}, T) + C_2(z, \tilde{z}, T) \int_0^t E(v)(t') dt',$$

for  $t \in [0, T]$ , where  $C_1(z, \tilde{z}, T)$  and  $C_2(z, \tilde{z}, T)$  satisfy

$$\begin{aligned} C_1(z, \tilde{z}, T) &\sim \|z\|_{L_{T,x}^{10}}^{10} + \|z\|_{L_T^\infty L_x^6}^6 + \|\tilde{z}\|_{L_{T,x}^6}^6 \leq \lambda(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}) < \infty, \\ C_2(z, \tilde{z}, T) &\sim 1 + \|z\|_{L_{T,x}^\infty}^2 + \|\langle \nabla \rangle^{s-\delta} \tilde{z}\|_{L_{T,x}^\infty} \leq \lambda(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}) < \infty. \end{aligned}$$

Finally, the energy bound (4.7) follows from Gronwall's inequality.  $\square$

## 5. DETERMINISTIC ANALYSIS OF THE PERTURBED NLW

In this section, we discuss the deterministic component of the proof of Theorem 1.5. Given a deterministic real-valued function  $f$ , we consider the Cauchy problem of the following perturbed defocusing quintic NLW:

$$\begin{cases} \partial_t^2 v - \Delta v + (v + f)^5 = 0 \\ (v, \partial_t v)|_{t=t_0} = (v_0, v_1). \end{cases} \quad (5.1)$$

In this section, we prove long time existence of solutions to (5.1) under some appropriate assumptions on  $f$ .

First, we briefly discuss the local well-posedness of (5.1). If one applies the Strichartz estimates (Lemma 2.1) and a simple fixed point argument to prove local well-posedness of (5.1) in the energy space, then the time of local existence depends on the profile of the initial data. This, however, can be upgraded to the following “good” local well-posedness result, where the time of local existence is characterized only in terms of the  $\dot{\mathcal{H}}^1$ -norm of the initial data  $(v_0, v_1)$  and the size of the perturbation  $f$ .

**Lemma 5.1** (Proposition 4.3 in [25]). *Let  $(v_0, v_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^3)$ . Then, there exists a function  $\tau : [0, \infty) \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , non-increasing in the first two arguments, such that if  $f$  satisfies the condition*

$$\|f\|_{L_t^5 L_x^{10}([t_0, t_0 + \tau_*])} \leq K \tau_*^\theta \quad (5.2)$$

for some  $K, \theta > 0$  and  $\tau_* \leq \tau = \tau(\|(v_0, v_1)\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)}, K, \theta) \ll 1$ , then there exists a unique solution  $(v, \partial_t v) \in C([t_0, t_0 + \tau_*]; \dot{\mathcal{H}}^1(\mathbb{R}^3))$  to (5.1).

Lemma 5.1 is the exact analogue on  $\mathbb{R}^3$  of Proposition 4.3 in [25]. Its proof is based on the global space-time bounds of solutions to the energy-critical defocusing NLW from [1, 29] and a perturbation argument, in particular, the long time perturbation lemma (Lemma 4.5 in [25]). See [25] for the details of the proof.

Given finite  $T \gg 1$ , our goal is to construct a solution to (5.1) on  $[0, T]$  for suitable  $f$ . If there is an a priori energy control:

$$\sup_{t \in [0, T]} \|(v(t), \partial_t v(t))\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)} < C(T) < \infty, \quad (5.3)$$

then Lemma 5.1 allows us to construct a solution  $v$  to (5.1) on  $[0, T]$ . Indeed, the analogue of Lemma 5.1 on  $\mathbb{R}^d$ ,  $d = 4, 5$ , was the main part of the deterministic analysis in [25]. It was then combined with the probabilistic a priori energy bound (1.16) to prove almost sure global well-posedness of the defocusing energy-critical NLW on  $\mathbb{R}^d$ ,  $d = 4, 5$ .

Our setting is slightly different; we do not assume an a priori energy control (5.3). Instead, we assume a *uniform* a priori energy control (see (5.6) below) on smooth approximating solutions  $v_N$  and construct a solution  $v$  to (5.1) on long time intervals (Proposition 5.2 below). In Section 6, we will combine this result with the uniform probabilistic a priori energy bound on smooth approximating solutions (Proposition 4.1) and prove almost sure global well-posedness of (1.1) below the energy space.

Given  $f \in L_{t, \text{loc}}^5 L_x^{10}$ , let  $f_N = \mathbf{P}_{\leq N} f$  for dyadic  $N \geq 1$ . Consider the following perturbed NLW:

$$\begin{cases} \partial_t^2 v_N - \Delta v_N + (v_N + f_N)^5 = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (0, 0). \end{cases} \quad (5.4)$$

The following proposition is the main result of this section.

**Proposition 5.2.** *Let  $f, f_N$ , and  $v_N$  be as above. Given finite  $T > 0$ , assume that the following conditions hold:*

(i) *There exist  $K, \theta > 0$  such that*

$$\|f\|_{L_t^5 L_x^{10}(I \times \mathbb{R}^3)} \leq K |I|^\theta \quad (5.5)$$

*for any compact interval  $I \subset [0, T]$ .*

(ii) *For each dyadic  $N \geq 1$ , a solution  $v_N$  to (5.4) exists on  $[0, T]$  and satisfies the following uniform a priori energy bound:*

$$\sup_N \sup_{t \in [0, T]} \|(v_N(t), \partial_t v_N(t))\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)} < C_0(T) < \infty. \quad (5.6)$$

(iii) *There exists  $\alpha > 0$  such that*

$$\|f - f_N\|_{L_T^5 L_x^{10}} < C_1(T) N^{-\alpha} \quad (5.7)$$

*for all dyadic  $N \geq 1$ .*

Then, there exists a unique solution  $(v, \partial_t v) \in C([0, T]; \mathcal{H}^1(\mathbb{R}^3))$  to (5.1) with  $(v, \partial_t v)|_{t=0} = (0, 0)$ , satisfying

$$\sup_{t \in [0, T]} \|(v(t), \partial_t v(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} < 2C_0(T) < \infty. \quad (5.8)$$

*Proof.* Given  $T > 0$ , fix

$$\tau_0 := \tau(2C_0(T), K, \theta), \quad (5.9)$$

where  $\tau$  and  $C_0(T)$  are as in Lemma 5.1 and (5.6), respectively. Fix  $0 < \tau_* \leq \tau_0$  and divide the time interval  $[0, T]$  into  $O(\frac{T}{\tau_*})$ -many subintervals of length  $\tau_*$  and denote them by

$$I_j := [j\tau_*, (j+1)\tau_*] \cap [0, T],$$

$j = 0, 1, \dots, [\frac{T}{\tau_*}]$ . The basic idea of the proof is to iteratively apply Lemma 5.1 on each  $I_j$ , while controlling the growth of the  $\dot{\mathcal{H}}^1$ -norm of  $(v, \partial_t v)$  on  $I_j$ . In the following, various constants depend on  $K$ ,  $\theta$ , and  $\alpha$  in (5.5) and (5.7), but we suppress their dependence.

We start with a brief description of the properties of the solution  $v_N$  to (5.4). By (5.5) and (5.7), we have

$$\|f_N\|_{L_t^5 L_x^{10}(I \times \mathbb{R}^3)} \leq K|I|^\theta + C_1(T)N^{-\alpha}, \quad (5.10)$$

for any compact interval  $I \subset [0, T]$ . It follows from a slight modification of the proof of Proposition 4.3 in [25] that there exists  $N_1 = N_1(T, \|(v_0, v_1)\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)})$  such that an analogue of Lemma 5.1 holds for

$$\begin{cases} \partial_t^2 v_N - \Delta v_N + (v_N + f_N)^5 = 0 \\ (v_N, \partial_t v_N)|_{t=t_0} = (v_0, v_1) \end{cases} \quad (5.11)$$

as long as  $N \geq N_1$ . More precisely, there exists  $\tau_1 = \tau_1(\|(v_0, v_1)\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)}, K, \theta) \ll 1$  such that, if  $f_N$  satisfies (5.10) on  $I = [t_0, t_0 + \tilde{\tau}]$  for some  $0 < \tilde{\tau} \leq \tau_1$ , then there exists a unique solution  $(v_N, \partial_t v_N) \in C([t_0, t_0 + \tilde{\tau}]; \dot{\mathcal{H}}^1(\mathbb{R}^3))$  to (5.11).

Let  $\tau_2 := \tau_1(C_0(T), K, \theta)$ . Then, in view of (5.6), we can apply this observation iteratively on intervals  $\tilde{I}_k := [k\tau_2, (k+1)\tau_2]$ ,  $k = 0, 1, \dots, [\frac{T}{\tau_2}]$ , and define the solution  $v_N$  on the whole interval  $[0, T]$ . Moreover, it follows from the proof of Proposition 4.3 in [25] that there exist  $\eta \ll 1$  and  $J(C_0(T)) \in \mathbb{N}$  such that we can decompose the time interval interval  $\tilde{I}_k$  into  $J'_k$ -many subintervals  $\tilde{I}_{k,\ell}$  for some  $J'_k \leq J(C_0(T))$  with the property that

$$\|v_N\|_{L_{\tilde{I}_{k,\ell}}^5 L_x^{10}} \leq 4\eta, \quad (5.12)$$

for all  $k = 0, 1, \dots, [\frac{T}{\tau_2}]$  and  $j = 1, 2, \dots, J'_k$ .

We now begin the construction of the solution  $v$  to (5.1). Since  $(v, \partial_t v)|_{t=0} = (0, 0)$ , we have  $\|(v(0), \partial_t v(0))\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)} = 0 \leq 2C_0(T)$ . Thus, Lemma 5.1 guarantees the existence of  $v$  on  $I_0 := [0, \tau_*] \subset [0, \tau_0]$ . In particular, we have  $(v, \partial_t v) \in C(I_0; \dot{\mathcal{H}}^1(\mathbb{R}^3))$ . Moreover, it follows from the proof of Proposition 4.3 in [25] that there exists a decomposition of the time interval  $I_0$  into  $J_0$ -many subintervals  $I_{0,m}$ , with the property that

$$\|v\|_{L_{I_{0,m}}^5 L_x^{10}} \leq 4\eta, \quad (5.13)$$

for all  $m = 1, 2, \dots, J_0$ .

Next, consider the following decomposition of  $I_0$ :

$$I_0 = \bigcup_{k,\ell,m} \{I_{0,k,\ell,m} := I_{0,m} \cap \tilde{I}_{k,\ell} : \tilde{I}_{k,\ell} \cap I_0 \neq \emptyset\}.$$

Note that this decomposition contains at most  $(\lceil \frac{T}{\tau_2} \rceil + 1)J(C_0(T))J_0$  subintervals. For notational simplicity, let  $I := I_{0,k,\ell,m}$ ,  $t_0 := \min I_{0,k,\ell,m}$ , and  $w_N := (v - v_N, \partial_t v - \partial_t v_N)$ . Then, it follows from (5.13), (5.12), (5.5), (5.10), and making  $\tau_0$  smaller, if necessary, that there exists  $N_2 = N_2(T) \geq N_1$  such that

$$\begin{aligned} & \|v\|_{L_I^5 L_x^{10}}^4 + \|v_N\|_{L_I^5 L_x^{10}}^4 + \|f\|_{L_I^5 L_x^{10}}^4 + \|f_N\|_{L_I^5 L_x^{10}}^4 \\ & \lesssim \eta^4 + K^4 \tau_0^{4\theta} + [C_1(T)N^{-\alpha}]^4 \ll 1, \end{aligned} \quad (5.14)$$

for all  $N \geq N_2$ . Then, by Lemma 2.1 and (5.14), we have

$$\begin{aligned} & \|w_N\|_{L_I^\infty \dot{\mathcal{H}}^1} + \|v - v_N\|_{L_I^5 L_x^{10}} \\ & \leq C_2 \|w_N(t_0)\|_{\dot{\mathcal{H}}^1} + \frac{1}{2} \|v - v_N\|_{L_I^5 L_x^{10}} + \frac{1}{2} \|f - f_N\|_{L_I^5 L_x^{10}}. \end{aligned} \quad (5.15)$$

Hence, it follows from (5.7) and (5.15) that

$$\|w_N\|_{L_I^\infty \dot{\mathcal{H}}^1} + \|v - v_N\|_{L_I^5 L_x^{10}} \leq C_3(T) (\|w_N(t_0)\|_{\dot{\mathcal{H}}^1} + N^{-\alpha}) \quad (5.16)$$

for all  $N \geq N_2$ . Then, applying (5.16) with  $w_N(0) = 0$  and (4.6) on all the subintervals  $I = I_{0,k,\ell,m}$  in an iterative manner, we obtain

$$\|w_N\|_{L_{I_0}^\infty \dot{\mathcal{H}}^1} \leq T(C_3(T) + 1)^{(\lceil \frac{T}{\tau_2} \rceil + 1)J(C_0(T))J_0} N^{-\alpha}. \quad (5.17)$$

Then, it follows from (5.17) and (5.6) that there exists  $N_3 = N_3(T, \tau_2) \geq N_2$  such that

$$\|(v, \partial_t v)\|_{L_{I_0}^\infty \dot{\mathcal{H}}^1} \leq C_0(T) + T(C_3(T) + 1)^{(\lceil \frac{T}{\tau_2} \rceil + 1)J(C_0(T))J_0} N^{-\alpha} \leq 2C_0(T) \quad (5.18)$$

for all  $N \geq N_3$ . This in particular implies that

$$\|(v(\tau_*), \partial_t v(\tau_*))\|_{\dot{\mathcal{H}}^1(\mathbb{R}^3)} \leq 2C_0(T).$$

Thus, we can apply Lemma 5.1 and construct a solution  $(v, \partial_t v) \in C(I_1; \dot{\mathcal{H}}^1)$ . Moreover, it follows from the proof of Proposition 4.3 in [25] that there exists a decomposition of the time interval  $I_1$  into  $J(2C_0(T))$ -many subintervals  $I_{1,m}$  with the property that

$$\|v\|_{L_{I_{1,m}}^5 L_x^{10}} \leq 4\eta,$$

for all  $m = 1, 2, \dots, J(2C_0(T))$ . Arguing as before, there exists  $N_4 = N_4(T, \tau_2) \geq N_3$  such that

$$\begin{aligned} \|(v, \partial_t v)\|_{L_{I_1}^\infty \dot{\mathcal{H}}^1} & \leq C_0(T) + T(C_3(T) + 1)^{(\lceil \frac{T}{\tau_2} \rceil + 1)J(C_0(T))(J_0 + J(2C_0(T)))} N^{-\alpha} \\ & \leq 2C_0(T) \end{aligned} \quad (5.19)$$

for all  $N \geq N_4$ . In view of (5.19), we can clearly apply Lemma 5.1 and extend the solution  $v$  onto  $I_2$ .

Arguing inductively, we can extend the solution  $v$  onto the entire interval  $[0, T]$ . Furthermore, there exists  $N_0 = N_0(T, \tau_2, \tau_*) \in \mathbb{N}$  such that

$$\begin{aligned} \sup_{t \in [0, T]} \|(v(t), \partial_t v(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} &\leq C_0(T) + T(C_3(T) + 1)^{([\frac{T}{\tau_2}] + 1)J(C_0(T))\{J_0 + [\frac{T}{\tau_*}]J(2C_0(T))\}} N^{-\alpha} \\ &< 2C_0(T), \end{aligned}$$

for all  $N \geq N_0$ . Hence, the energy estimate (5.8) is also satisfied on  $[0, T]$ .  $\square$

**Remark 5.3.** (i) The condition (5.6) can be relaxed as follows; it suffices to assume

$$\sup_{t \in [0, T]} \|(v_{N_0}(t), \partial_t v_{N_0}(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} < C_0(T) < \infty,$$

for some  $N_0 = N_0(T, C_0(T)) \gg 1$ .

(ii) The proof of Proposition 5.2 shows that the hypothesis (5.5) can also be relaxed. Let  $\tau_* \leq \tau_0$ , where  $\tau_0$  is as in (5.9). Then, by setting  $I_j = [j\tau_*, (j+1)\tau_*] \cap [0, T]$ , it suffices to assume that there exist  $K, \theta > 0$  such that

$$\|f\|_{L_{I_j}^5 L_x^{10}} \leq K|I_j|^\theta \ll 1$$

for all  $j = 0, \dots, [\frac{T}{\tau_*}]$ .

## 6. ALMOST SURE GLOBAL EXISTENCE

In this section, we present the proof of Theorem 1.5. Note that Theorem 1.5 follows once we prove the following ‘almost’ almost sure global well-posedness for (1.1). See [13, 5] for details on this reduction.

**Proposition 6.1** (‘Almost’ almost sure global well-posedness). *Let  $s \in (\frac{1}{2}, 1)$  and  $T \geq 1$ . Given  $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^3)$ , let  $(u_0^\omega, u_1^\omega)$  be the Wiener randomization defined in (1.10), satisfying (1.12). Then, given any  $T, \varepsilon > 0$ , there exists  $\Omega_{T, \varepsilon} \subset \Omega$  such that*

- (i)  $P(\Omega_{T, \varepsilon}^c) < \varepsilon$ ,
- (ii) For any  $\omega \in \Omega_{T, \varepsilon}$ , there exists a unique solution  $u^\omega$  to (1.1) on  $[0, T]$  with  $(u^\omega, \partial_t u^\omega)|_{t=0} = (u_0^\omega, u_1^\omega)$  in the class:

$$(S(t)(u_0^\omega, u_1^\omega), \partial_t S(t)(u_0^\omega, u_1^\omega)) + C([0, T]; \mathcal{H}^1(\mathbb{R}^3)) \subset C([0, T]; \mathcal{H}^s(\mathbb{R}^3)).$$

- (iii) For any  $\omega \in \Omega_{T, \varepsilon}$ , the following probabilistic energy bound holds for the nonlinear part  $v^\omega$  of the solution  $u^\omega$ :

$$\sup_{t \in [0, T]} \|(v^\omega(t), \partial_t v^\omega(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} < C(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}).$$

The main ingredients of the proof of Proposition 6.1 are the probabilistic uniform energy bound on approximating solutions (Proposition 4.1) and the deterministic long time existence for the perturbed NLW (5.1) (Proposition 5.2).

*Proof.* Given  $(u_0^\omega, u_1^\omega)$ , let  $z^\omega$  and  $z_N^\omega$  be as in (1.14) and (4.2), respectively. With  $\alpha \in (0, s]$ , set

$$M = M(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^\alpha}) \sim T^{\frac{6}{5}} \left( \log \frac{1}{\varepsilon} \right)^{\frac{1}{2}} \|(u_0, u_1)\|_{\mathcal{H}^\alpha}.$$

Then, defining  $\Omega_1 = \Omega_1(T, \varepsilon)$  by

$$\Omega_1 := \{\omega \in \Omega : \|\langle \nabla \rangle^\alpha z^\omega\|_{L_T^5 L_x^{10}} \leq M\},$$

it follows from Lemma 3.2 (ii) that

$$P(\Omega_1^c) < \frac{\varepsilon}{3}. \quad (6.1)$$

Moreover, for each  $\omega \in \Omega_1$ , we have

$$\|z^\omega - z_N^\omega\|_{L_T^5 L_x^{10}} \leq N^{-\alpha} \|\langle \nabla \rangle^\alpha z^\omega\|_{L_T^5 L_x^{10}} \leq MN^{-\alpha}. \quad (6.2)$$

Given dyadic  $N \geq 1$ , apply Proposition 4.1 and construct  $\Omega_2(N) := \tilde{\Omega}_{N,T,\frac{\varepsilon}{3}}$  with

$$P(\Omega_2(N)^c) < \frac{\varepsilon}{3} \quad (6.3)$$

such that

$$\sup_{t \in [0, T]} \|(v_N^\omega(t), \partial_t v_N^\omega(t))\|_{\mathcal{H}^1} \leq C_0(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s}) < \infty, \quad (6.4)$$

for each  $\omega \in \Omega_2(N)$ . The main point here is that  $C_0 = C_0(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s})$  can be chosen independent of  $N$ .

Fix  $K = \|(u_0, u_1)\|_{\mathcal{H}^0}$  and  $\theta = \frac{1}{10}$  in the following. Let  $\tau_* \leq \tau_0$  to be chosen later, where  $\tau_0 = \tau(2C_0(T), K, \theta)$  is as in (5.9). By writing  $[0, T] = \bigcup_{j=0}^{\lfloor T/\tau_* \rfloor} I_j$  with  $I_j = [j\tau_*, (j+1)\tau_*] \cap [0, T]$ , define  $\Omega_3$  by

$$\Omega_3 := \left\{ \omega \in \Omega : \|z^\omega\|_{L_{I_j}^5 L_x^{10}} \leq K |I_j|^\theta, j = 0, \dots, \lfloor \frac{T}{\tau_*} \rfloor \right\}. \quad (6.5)$$

Then, by Lemma 3.2 with  $|I_j| \leq \tau_*$ , we have

$$P(\Omega_3^c) \leq \sum_{j=0}^{\lfloor \frac{T}{\tau_*} \rfloor} P\left(\|z^\omega\|_{L_{I_j}^5 L_x^{10}} > K |I_j|^\theta\right) \lesssim \frac{T}{\tau_*} \exp\left(-\frac{c}{T^2 \tau_*^{\frac{1}{5}}}\right).$$

By making  $\tau_*$  smaller if necessary,

$$\lesssim \frac{T}{\tau_*} \exp\left(-\frac{c}{2T^2 \tau_*^{\frac{1}{5}}}\right) = T \exp\left(-\frac{c}{2T^2 \tau_*^{\frac{1}{5}}}\right).$$

Hence, by choosing  $\tau_* = \tau_*(T, \varepsilon)$  sufficiently small, we have

$$P(\Omega_3^c) < \frac{\varepsilon}{3}. \quad (6.6)$$

Let  $\Omega_{T,\varepsilon} := \Omega_1 \cap \Omega_2(N_0) \cap \Omega_3$ , where  $N_0$  is to be chosen later. Then, from (6.1), (6.3), and (6.6), we have

$$P(\Omega_{T,\varepsilon}^c) < \varepsilon.$$

By choosing  $N_0 = N_0(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s}) \gg 1$ , it follows from Proposition 5.2 and Remark 5.3 with (6.2), (6.4), and (6.5), that there exists a solution  $v^\omega$  to (1.15) on  $[0, T]$  for each  $\omega \in \Omega_{T,\varepsilon}$ . Hence, for  $\omega \in \Omega_{T,\varepsilon}$ , there exists a solution  $u^\omega = z^\omega + v^\omega$  to (1.1) on  $[0, T]$ . Moreover, the following estimate holds:

$$\sup_{t \in [0, T]} \|(v^\omega(t), \partial_t v^\omega(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} < 2C_0(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)}) < \infty.$$



This completes the proof of Proposition 6.1 and hence the proof of Theorem 1.5.  $\square$

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TADAHIRO OH, SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, AND THE MAXWELL INSTITUTE FOR THE MATHEMATICAL SCIENCES, JAMES CLERK MAXWELL BUILDING, THE KING'S BUILDINGS, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, UNITED KINGDOM

*E-mail address:* hiro.oh@ed.ac.uk

OANA POCOVNICU, DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON RD., PRINCETON, NJ 08544-1000, USA, AND DEPARTMENT OF MATHEMATICS, HERIOT-WATT UNIVERSITY AND THE MAXWELL INSTITUTE FOR THE MATHEMATICAL SCIENCES, EDINBURGH, EH14 4AS, UNITED KINGDOM

*E-mail address:* o.pocovnicu@hw.ac.uk